Assignment A1

Instructions

Due ~~Thurs Week 8, Thurs 15th March~~ Mon Week 9, Mon 19th March, by 1700

Submission is via Learn. You should submit your assignment using "Attach Files" under "Assignment Submission". You should submit *only the requested files*: `poisson_L.m`, `solve_L.m`, `solve_L_T.m`, `solve_poisson.m`, `test_poisson.m`, and `runge.m`. These should be function or script .m files that can be used *without modification.*

Marks are given in bold immediately following each question. The assignment is marked out of 35, and counts for 30% of the course.

Your code should be extensively commented, with the functionality of each line of code explained with a comment. All lines of code *except* for `end` statements, must include such a comment. See the provided function .m file `is_prime_commented.m` for an example of the level of commenting expected. Solutions to coding related questions which lack explanatory comments will receive no more than half marks. For this assignment this applies to all questions except for question 2.2.

Note that this is an individual assignment and is primarily summative in nature. Please refer to the lecture 1 slides "AcademicMisconduct-201718.pdf", and the information in the "Assessment" folder on Learn, for academic misconduct advice and policies.

1 Forward and backward substitution

Consider the second order linear inhomogeneous ordinary differential equation

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} = e^x \sin\left(2\pi x\right), \tag{1}$$

with boundary conditions

$$\Psi\left(x=0\right) = 0, \tag{2a}$$
$$\Psi\left(x=1\right) = 0. \tag{2b}$$

The second derivative can be approximated via

$$\left.\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2}\right|_{x=x_i} \approx \frac{\psi_{i-1} - 2\psi_i + \psi_{i+1}}{\left(\Delta x\right)^2}, \tag{3}$$

where $\psi_i \approx \Psi\left(x = x_i\right)$ is some approximation for the solution to the differential equation, $x_i = i\Delta x$, and where $\Delta x = 1/N$ for some positive integer N. This suggests that one replace the ordinary differential equation (1) with the *finite difference* approximation

$$\frac{\psi_{i-1} - 2\psi_i + \psi_{i+1}}{\left(\Delta x\right)^2} = e^{x_i} \sin\left(2\pi x_i\right) \qquad \text{for } i \in \left\{1, 2, \ldots, N-1\right\}, \tag{4}$$

with boundary conditions

$$\psi_0 = 0, \tag{5a}$$
$$\psi_N = 0. \tag{5b}$$

Solving the $(N-1)$ equations in (4) is equivalent to solution of the linear system

$$Az = b, \tag{6}$$

where A is a tri-diagonal matrix of size $(N-1) \times (N-1)$ with

$$A = \begin{pmatrix} 2 & -1 & 0 & & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & & 0 & 0 & 0 \\ & \vdots & & & & \vdots & \\ 0 & 0 & 0 & & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & & 0 & -1 & 2 \end{pmatrix}, \tag{7}$$

i.e.

$$A_{i,j} = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } j = i-1 \text{ or } j = i+1 \\ 0 & \text{otherwise} \end{cases}, \tag{8}$$

and z and b are length $(N-1)$ column vectors with

$$z = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_{N-1} \end{pmatrix}, \qquad b = -(\Delta x)^2 \begin{pmatrix} e^{x_1} \sin(2\pi x_1) \\ e^{x_2} \sin(2\pi x_2) \\ \vdots \\ e^{x_{N-1}} \sin(2\pi x_{N-1}) \end{pmatrix}, \tag{9}$$

i.e.

$$z_i = \psi_i, \tag{10a}$$
$$b_i = -\frac{1}{N^2} e^{i/N} \sin\left(\frac{2\pi i}{N}\right). \tag{10b}$$

Note that here i is an integer element index (i.e. is *not* imaginary).

1.1 It can be shown that

$$LL^T = A, \tag{11}$$

where L is a square matrix of size $(N-1) \times (N-1)$, and is bi-diagonal and lower triangular with elements

$$L_{i,j} = \begin{cases} +\sqrt{\frac{i+1}{i}} & \text{if } i = j \\ -\sqrt{\frac{i-1}{i}} & \text{if } j = i-1 \\ 0 & \text{otherwise} \end{cases}. \tag{12}$$

Write a function .m file `poisson_L.m` defining a function which accepts a single integer argument `N`, and returns the matrix L. **5 marks**

1.2 Write a function .m file `solve_L.m` defining a function which accepts a single argument `c`, equal to a column vector c of length $(N-1)$, and returns a vector y where

$$Ly = c. \tag{13}$$

You may assume that $N \geq 2$.

Your solution should use forward substitution, and should *not* construct the matrix L explicitly (e.g. should *not* make use of `poisson_L.m`). **5 marks**

1.3 Write a function .m file `solve_L_T.m` defining a function which accepts a single argument `c`, equal to a column vector c of length $(N-1)$, and returns a vector y where

$$L^T y = c. \tag{14}$$

You may assume that $N \geq 2$.

Your solution should use backward substitution, and should *not* construct the matrix L^T explicitly (e.g. should *not* make use of `poisson_L.m`). Note that L^T has elements

$$L^T_{i,j} = \begin{cases} +\sqrt{\frac{i+1}{i}} & \text{if } i = j \\ -\sqrt{\frac{i}{i+1}} & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}. \tag{15}$$

5 marks

1.4 Write a function .m file `solve_poisson.m` defining a function which accepts a single input argument `N`, equal to N, and returns a column vector z where $z^T = (\psi_1, \psi_2, \ldots, \psi_{N-1})$ is the solution to (6). You may assume that $N \geq 2$. You should make use of your solutions to questions 1.2 and 1.3.

Your function should solve the equation by considering the solution of two sub-problems

$$Ly = b, \tag{16a}$$
$$L^T z = y, \tag{16b}$$

where b is the vector on the right-hand-side of equation (6).

3 marks

1.5 Write a script .m file `test_poisson.m` which plots the numerical solution defined by $\{\psi_0, \psi_1, \ldots, \psi_N\}$ for $N = 32$. You should use appropriate commands to format the plot, including the addition of appropriate labelling. **3 marks**

2 Interpolation

For further background on "Runge's example" see section 4.2 of "Introduction to Scientific Computing Using Matlab", I. Gladwell, J. G. Nagy, and W. F. Ferguson, Jr, 2011.

Consider the function

$$F(x) = \frac{1}{1+x^2}, \tag{17}$$

on the interval $x \in [-10, 10]$. Evaluating this at 9 evenly spaced points yields coordinates

$$x_i = \frac{5}{2}(i-1) - 10 \quad \text{for } i \in \{1, 2, \ldots, 9\}, \tag{18a}$$
$$f_i = F(x_i) \quad \text{for } i \in \{1, 2, \ldots, 9\}. \tag{18b}$$

2.1 Write a script .m file `runge.m` which displays a scatter plot of the (x_i, f_i) coordinates, and also plots

- The function $F(x)$.

- The degree 8 interpolating polynomial fitting through the (x_i, f_i).
- A degree 4 least squares fitting polynomial fitting the (x_i, f_i).
- A cubic spline fitting through the (x_i, f_i), generated using the `spline` function.

The script should display all the plots in a single figure window. You should use appropriate commands to format the plot, including the addition of a legend.

6 marks

2.2 Add a discussion, in the form of a code comment in `runge.m` of not more than 30 lines in length, comparing the different fits generated in question 2.1, and describing the key properties of the fitting functions.

5 marks

3 Clarity and scholarship

You code should be neatly presented and easy to understand You should include any relevant references to textbooks or other materials in your explanatory comments.

3 marks